Guide to Proofs on Sets

Richard Feynman, one of the greatest physicists of the twentieth century, gave a famous series of lectures on physics while a professor at Caltech. Those lectures have been recorded for posterity and are legendary for their blend of intuitive and mathematical reasoning. One of my favorite quotes from these lectures comes early on, when Feynman talks about the atomic theory of matter. Here's a relevant excerpt:

> If, in some cataclysm, all of scientific knowledge were to be destroyed, and only one sentence passed on to the next generations of creatures, what statement would contain the most information in the fewest words? I believe it is the atomic hypothesis (or the atomic fact, or whatever you wish to call it) that ***all things are made of atoms*** [. ...] In that one sentence, you will see, there is an enormous amount of information about the world, if just a little imagination and thinking are applied.

This idea argues for a serious shift in perspective about how to interpret properties of objects in the physical world. Think about how, for example, you might try to understand why steel is so much stronger and tougher than charcoal and iron. If you don't have atomic theory, you'd probably think about what sorts of "inherent qualities" charcoal and iron each possess, and how those qualities interacting with one another would give rise to steel's strength. That's how you'd think about things if you were an alchemist. But with atomic theory, you could ask questions like "how are the atoms in a sample of steel arranged?," "what happens to an assortment of atoms in a metal when the metal bends or deforms?," and "how would throwing some carbon atoms in the mix change that?" That's how you'd think about things if you were a modern chemist or materials scientist.

I would argue that if you have a single guiding principle for how to mathematically reason about sets, it would be this one:

All sets are made of elements, and they're completely defined by their elements.

Think of this as the analog of atomic theory for sets. Although professional mathematicians don't use this term, I like to call it the ***elemental theory of sets***, since it emphasizes the core idea that sets are made of elements, and that understanding how sets behave really boils down to understanding how their elements behave.

To formalize your intuition about sets and how they behave – and to build up better predictions for how sets will interact with one another – you'll want to shift your thinking from a holistic "$A \cup B$ represents the set you get when you combine everything from A and B together" to a more precise "$x \in A \cup B$ if and only if $x \in A$ or $x \in B$." That change in perspective – from the properties of the set as a whole to properties of the individual elements of those sets – will be a key theme throughout this course. It's not necessarily the most natural perspective to adopt, but once you've learned to think about things this way you'll get a much deeper understanding for how sets behave.

The rest of this handout explores how to think about things this way, as well as how to use that perspective to write formal proofs.

A First Running Example

In the upcoming sections, we're going to see how to reason rigorously about sets and set theory. Rather than doing that in the abstract, we'll focus on a specific, concrete example.

Consider the following theorem:

Theorem: For any sets A, B, C, D, and E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, we have $A \subseteq D \cup E$.

Although there are a ton of variables here, this result isn't as scary as it might look. Before moving on, take a minute to think through what's going on here. Draw some pictures. Try out some examples. Can you get a handle on what's going on here? As with any mathematical proof, the first step is to try to get a handle on how all the pieces move. (Having trouble drawing this? *Ask about it on Piazza!*)

At this point you might have a sense for why this theorem is true. You might also have no intuition for what's going on here. And that's fine! Because what we're going to do now is to see how we might tackle writing a proof. It might seem weird to approach writing a proof of a result when you still haven't figured out how everything fits together. And that's a good intuition to have. However, in many cases, the act of sitting down and trying to figure out what the proof might look like might give you some clarity into what questions you should be trying to ask and where you should focus your efforts.

To begin with, even though this theorem involves a bunch of variables all related in some weird ways, and even though you might not have an idea of where this proof is going to go, you can still at the very least set up the first sentence. As we saw in lecture, there are a number of little mini "proof templates" that you can use to focus your efforts. Here, we're trying to prove a universally-quantified statement (*"for any sets ... where ..."*), and as you saw in class, there's a nice template for starting this one off:

> ***Theorem:*** For any sets A, B, C, D, and E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, we have $A \subseteq D \cup E$.
>
> ***Proof:*** Consider any sets A, B, C, D, and E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$. We will prove that $A \subseteq D \cup E$. [the rest of the proof goes here.]

In other words, we're beginning with a ton of little assumptions, and we have a single goal that we need to prove (namely, $A \subseteq D \cup E$). So now the question is how we go about doing that.

Notice that, up to this point, we haven't actually needed to know anything about sets or set theory. We could just as easily have replaced the word "set" with "pizkwat," the symbol "$\subseteq$" with the word "gloobah," and the symbol $\cup$ with "zyzzyzyplyx," and the proof setup would have looked the same. (Or rather, it would have had the same structure, but looked much sillier.) To make actual progress here, we need to know how to reason about subsets, unions, and the like. So let's see how to do that.

Reasoning About Subsets

The subset-of relation $\subseteq$ is one of the most fundamental relations we'll explore in set theory. As a reminder, formally speaking, we say that $S \subseteq T$ if every element of S is also an element of T. If you'll notice, the statement "every element of S is also an element of T" is a universal statement – it says that for each object of some type ("every element of S") has some other property ("is also an element of T.") And as you saw on Wednesday, there's a nice technique for proving universal statements that involves making arbitrary choices.

Putting this together, we have the following:

☞ **To prove $S \subseteq T$, pick an arbitrary $x \in S$, then prove that $x \in T$.** ☜

Using this template, we can continue the proof that we set up on the previous page. When we left off, we said we needed to prove $A \subseteq D \cup E$. And hey! We just developed a template for that. Let's use it!

Theorem: If $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, then $A \subseteq D \cup E$.

Proof: Consider any sets A, B, C, D, and E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$. We will prove that $A \subseteq D \cup E$. To do so, pick an arbitrary $x \in A$. We will prove that $x \in D \cup E$. [*the rest of the proof goes here.*]

An important detail here: ***this proof introduces a new variable x.*** The statement of the theorem purely relates A, B, C, D, and E to one another. It says nothing whatsoever about anything named x. Think back to the discussion of atomic theory and steel. Asking why steel is so strong requires you to steer the conversation away from "steel in general" and toward "individual atoms inside of a piece of steel." The original question – why is steel strong? – doesn't concern individual steel atoms, but *answering* that question requires you to think about things that way. Similarly, although there is no variable x in the original theorem, *proving* that theorem requires us to reason about elements of the sets. So just as atomic theory means "any discussion of fundamental properties of matter probably requires you to talk about atoms," the elemental theory of sets means "any discussion of sets will probably require you to introduce new variables to talk about individual elements."

How do we proceed from here? It's not immediately clear, but we can use some of the information we have. For example, we know that $A \subseteq B \cup C$, and we know that $x \in A$. We can combine these pieces of information together given the following principle:

☞ **If you know $x \in S$ and $S \subseteq T$, you can conclude $x \in T$.** ☜

This follows from how subsets are defined. If $S \subseteq T$, then every element of S is an element of T, and so in particular because x is an element of S, we can say that x is an element of T. We can put that into practice here:

Theorem: If $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, then $A \subseteq D \cup E$.

Proof: Consider any sets A, B, C, D, and E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$. We will prove that $A \subseteq D \cup E$. To do so, pick an arbitrary $x \in A$. We will prove that $x \in D \cup E$.

Since we know $x \in A$ and $A \subseteq B \cup C$, we see that $x \in B \cup C$. [*the rest of the proof goes here.*]

A detail to point out before we move on: notice that the way that we interact with the $\subseteq$ relation in a proof differs based on whether we are ***proving*** that one set is a subset of another or whether we are ***using*** the fact that one set is a subset of another. That will be unifying theme throughout the entire quarter, and you'll see this come up in the rest of this handout. In the first paragraph, we set up a ***proof*** that $A \subseteq D \cup E$ by picking an arbitrary $x \in A$. In the second, we ***used*** the fact that $A \subseteq B \cup C$ to conclude that $x \in B \cup C$. ***Proving*** that one set is a subset of another introduces a new variable; ***using*** the fact that one set is a subset of the other lets us conclude new things about existing variables.

Reasoning About Set Combinations

You probably have a good intuition for unions, intersections, and the like from your lived experience. The union of the set of all your TAs and your classmates represents the set of people you're mostly like to interact with in a given course. The intersection of the set of people you admire and the set of people who admire you represents the set of people you probably should consider becoming friends with. And so on.

But in the elemental theory of sets, we have to ask – what exactly makes up the sets $S \cup T$, $S \cap T$, $S \Delta T$, etc.? After all, sets are formally defined by their elements. And for that, we need these definitions:

$$S \cup T = \{ x \mid x \in S \text{ or } x \in T \text{ (or both)} \} \qquad S \cap T = \{ x \mid x \in S \text{ and } x \in T \}$$

$$S - T = \{ x \mid x \in S \text{ and } x \notin T \} \qquad S \Delta T = \{ x \mid \text{either } x \in S \text{ and } x \notin T, \text{ or } x \notin S \text{ and } x \in T \}$$

These are the definitions of these terms. It's good to know these definitions when you're thinking about how these sets operate. But in the context of proofwriting, you'll likely need to *use* these definitions in the following way:

If you know $x \in S \cup T$, you can conclude $x \in S$ or $x \in T$.

If you know $x \in S \cap T$, you can conclude $x \in S$ and $x \in T$.

If you know $x \in S - T$, you can conclude $x \in S$ and $x \notin T$.

If you know $x \in S \Delta T$, you can conclude either $x \in S$ and $x \notin T$, or $x \notin S$ and $x \in T$.

Let's jump back to the proof we're working through. We know that $x \in B \cup C$. Given what we just saw above, we can use that to conclude that $x \in B$ or $x \in C$. Let's use that to our advantage:

> ***Theorem:*** If $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, then $A \subseteq D \cup E$.
>
> ***Proof:*** Consider any sets A, B, C, D, and E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$. We will prove that $A \subseteq D \cup E$. To do so, pick an arbitrary $x \in A$. We will prove that $x \in D \cup E$.
>
> Since we know $x \in A$ and $A \subseteq B \cup C$, we see that $x \in B \cup C$. This in turn tells us that $x \in B$ or $x \in C$. [the rest of the proof goes here.]

We're making some progress here, because this lets us use some of the facts from our proof setup that we haven't touched yet. Specifically, we've been holding onto the fact that $B \subseteq D$ and that $C \subseteq E$, and here we're confronted with the fact that either $x \in B$ or $x \in C$. Using what we saw in the previous section about subsets, that means that we can potentially make a lot more progress here. The challenge is that we can't say for certain whether $x \in B$ or $x \in C$ – that might depend on x, B, and C. But that's not a problem – that's the sort of thing a proof by cases was meant for!

Here's how we might continue from the previous section using both a proof by cases and our knowledge of how B, C, D, and E relate:

Theorem: If $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, then $A \subseteq D \cup E$.

Proof: Consider any sets A, B, C, D, and E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$. We will prove that $A \subseteq D \cup E$. To do so, pick an arbitrary $x \in A$. We will prove that $x \in D \cup E$.

Since we know $x \in A$ and $A \subseteq B \cup C$, we see that $x \in B \cup C$. This in turn tells us that $x \in B$ or $x \in C$. We will therefore proceed by cases:

Case 1: $x \in B$. Then since $x \in B$ and $B \subseteq D$, we see that $x \in D$.

Case 2: $x \in C$. Then since $x \in C$ and $C \subseteq E$, we see that $x \in E$.

[*the rest of the proof goes here.*]

This is looking a lot better.

Our ultimate goal is to prove that $x \in D \cup E$. And based on where we are now, it seems like that goal is in sight! We know that $x \in D$ or that $x \in E$. And intuitively, that seems like that should be enough to conclude that $x \in D \cup E$, since, after all, $D \cup E$ is what you get when you take all the elements of D and all the elements of E and combine them together.

On the previous page, we saw how you could use the formal definitions of the set combination operators to go from knowledge that $x \in S \cup T$ to the conclusion that $x \in S$ or $x \in T$. In other words, if we already happen to know that an object is an element of a set union, we can use that to learn something about how that object connects with the individual sets that make up that union. But what about the other direction? What do we have to do to show that an object is an element of the union of two sets? For that, we can use this handy table:

To prove $x \in S \cup T$, prove that $x \in S$ or that $x \in T$.

To prove $x \in S \cap T$, prove that $x \in S$ and $x \in T$.

To prove $x \in S - T$, prove that $x \in S$ and $x \notin T$.

To prove that $x \in S \Delta T$, prove that $x \in S$ and $x \notin T$, or that $x \notin S$ and $x \in T$.

With this in mind, we can finish our proof! In each case, we learn that x belongs to one of the D or E, and so we can conclude that it always belongs to $D \cup E$.

Theorem: If $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, then $A \subseteq D \cup E$.

Proof: Consider any sets A, B, C, D, and E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$. We will prove that $A \subseteq D \cup E$. To do so, pick an arbitrary $x \in A$. We will prove that $x \in D \cup E$.

Since we know $x \in A$ and $A \subseteq B \cup C$, we see that $x \in B \cup C$. This in turn tells us that $x \in B$ or $x \in C$. We will therefore proceed by cases:

Case 1: $x \in B$. Then since $x \in B$ and $B \subseteq D$, we see that $x \in D$.

Case 2: $x \in C$. Then since $x \in C$ and $C \subseteq E$, we see that $x \in E$.

Collectively, these cases show that $x \in D$ or that $x \in E$. Therefore, we see that $x \in D \cup E$, as required. ■

And that's a wrap! Now, look back over this proof. Notice that this proof very heavily uses the elemental perspective on sets. We don't talk about how these sets, in general, relate to one another. We focused on a single element x, went with x on a magical journey, and ended up reaching our desired conclusion.

Reasoning About Set Equality

What does it mean for two sets to be equal? This is addressed by the fancy-sounding *axiom of extensionality*, a term you are totally welcome to toss around at cocktail parties, which says the following:

☞ **Two sets S and T are equal ($S = T$) if $S \subseteq T$ and $T \subseteq S$** ☜

This definition of set equality lets you make the following conclusions in the case where you know two sets are equal to one another:

☞ **If $S = T$ and $x \in S$, you can conclude that $x \in T$.**
If $S = T$ and $x \notin S$, you can conclude that $x \notin T$. ☜

The definition of set equality gives us the following route for proving two sets are equal:

☞ **To prove that $S = T$, prove that $S \subseteq T$ and $T \subseteq S$.** ☜

This approach for proving that two sets are equal is sometimes called a ***proof by double inclusion***, though we generally won't refer to it by that name. You are welcome to toss that around at cocktail parties as well, though, if you so choose.

Another consequence of this theorem is the following conclusion that you can also draw from two sets being equal to one another:

☞ **If $S = T$, you can conclude that $S \subseteq T$ and $T \subseteq S$.** ☜

This comes up every now and then, though it's much more common to use the opposite direction of this theorem to prove that $S = T$ via $S \subseteq T$ and $T \subseteq S$.

Reasoning About Power Sets

The power set is a strange creature. It's a set made of other sets, it's a set that converts subset-of to element-of, and it's a set that's used to show that different magnitudes of infinity exist. The good news is that, provided that you go slowly and methodically and don't skip any steps, it's not too hard to manipulate power sets in proofs.

First, let's recap the formal definition of the power set. The power set of a set S is the set of all subsets of S:

$$\wp(S) = \{ T \mid T \subseteq S \}$$

If you haven't already done so, take a minute to read over that set-builder notation and to see if you can convince yourself why it says symbolically what we described in plain English right above it.

The above definition is wonderfully useful. For example, if you want to show that an object belongs to $\wp(S)$, you need to show that that object obeys the set-builder notation. Specifically:

☞ **To prove that $T \in \wp(S)$, prove that $T \subseteq S$.** ☜

You can also run this definition the other way:

☞ **If you know $T \in \wp(S)$, you can conclude $T \subseteq S$.** ☜

A Second Example

Now that you've seen at a high level how you can reason about set equality and power sets, let's work through an example that will employ all of those techniques. Specifically, let's work through this example, which comes from the Winter 2018 midterm exam:

Theorem: For any sets A and B, we have $A \cap B = A$ if and only if $A \in \wp(B)$.

This theorem touches on two concepts about sets we haven't explored yet: power sets and set equality. It's also a biconditional, which means that it's two proofs for the price of one!

As before, I'll invite you to think about why, exactly, this result is true. This would be a great time to draw some pictures and to try out examples. See if you can build an intuition for what's going on here. *(Having trouble? Ask a question about it on Piazza!)*

Let's take a minute to think about how we'd formally set up a proof of this result. Again, even if you don't have a good intuition for where we're going or why this is true, it can still be really helpful to, at a bare minimum, set the proof up so that you see what exactly it is that we'll need to demonstrate. Following the lead from lecture, since this is a biconditional statement, we'll set the proof up as two separate halves that will work in tandem with one another.

Theorem: For any sets A and B, we have $A \cap B = A$ if and only if $A \in \wp(B)$.

Proof: Let A and B be arbitrary sets. We will prove both directions of implication.

($\Rightarrow$) First, assume that $A \cap B = A$. We will prove that $A \in \wp(B)$. [*the rest of this proof goes here.*]

($\Leftarrow$) Next, let's assume that $A \in \wp(B)$. We will prove that $A \cap B = A$. [*the rest of this proof goes here.*]

If you aren't familiar with the use of these double arrow markers at the start of each section of the proof (the $\Leftarrow$ and $\Rightarrow$ symbols), they're just a nice way to signal to the reader where each section of the biconditional proof begins. Although we generally discourage using symbols like these, this is a fairly common convention and makes the organization of the proof much simpler. We could have alternatively split this proof apart into a pair of lemmas (smaller proofs that build into bigger ones), one for each direction of implication, but we figured that this would be a slightly easier way to do things.

To make progress at this point, we'll need to use our specific knowledge about how to reason about set equality and power sets. For example, in the first branch of this proof, we need to prove that $A \cap B = A$, and in the second branch we'll need to prove that $A \in \wp(B)$. What do we need to do to prove these statements? Well, looking back at the preceding page, we can see that

- we can prove two sets are equal by showing that they're each subsets of one another, and
- we can prove that an object belongs to $\wp(S)$ by showing that it's a subset of S.

We can use that to expand the above proof, as is shown here:

Theorem: For any sets A and B, we have $A \cap B = A$ if and only if $A \in \wp(B)$.

Proof: Let A and B be arbitrary sets. We will prove both directions of implication.

(⇒) First, let's assume that $A \cap B = A$. We will prove that $A \in \wp(B)$. To do so, we'll prove that $A \subseteq B$. [*the rest of this proof goes here.*]

(⇐) Next, assume that $A \in \wp(B)$. We will prove that $A \cap B = A$. To do so, we'll prove that $A \cap B \subseteq A$ and that $A \subseteq A \cap B$. [*the rest of this proof goes here.*]

We can work on either branch of this proof at this point, but for simplicity's sake, let's start with that top branch. Here, we're assuming that $A \cap B = A$, and our goal is to prove that $A \subseteq B$. So let's ask – how exactly do you prove that $A \subseteq B$? That's something we saw earlier – you pick an arbitrary element $x \in A$ and then prove that $x \in B$. So let's add that in:

Theorem: For any sets A and B, we have $A \cap B = A$ if and only if $A \in \wp(B)$.

Proof: Let A and B be arbitrary sets. We will prove both directions of implication.

(⇒) First, let's assume that $A \cap B = A$. We will prove that $A \in \wp(B)$. To do so, we'll prove that $A \subseteq B$ by picking an arbitrary $x \in A$ and showing that $x \in B$. [*the rest of this proof goes here.*]

(⇐) Next, assume that $A \in \wp(B)$. We will prove that $A \cap B = A$. To do so, we'll prove that $A \cap B \subseteq A$ and that $A \subseteq A \cap B$. [*the rest of this proof goes here.*]

The question now is how to actually establish this. We've gotten to this point purely by expanding out definitions and following templates. Here, we need to pause and see if we can find some connection.

In this top branch, we're operating under the assumption that $A \cap B = A$, and that's something we haven't relied on yet. So maybe that's a good place to look – especially since there doesn't seem to be anything else to really do here.

Now, notice that in this section we are *assuming* that $A \cap B = A$, so we don't need to *prove* $A \cap B = A$. Instead, we can rely on the fact that $A \cap B = A$. That means we could bring in any of the facts about set equality that we saw earlier: that $A \cap B \subseteq A$, that $A \subseteq A \cap B$, that any $x \in A \cap B$ also satisfies $x \in A$, or that any $x \in A$ satisfies $x \in A \cap B$. How do we decide which of these paths to go down? Well, a priori, there's no reason to suspect that any one of them would pan out over the others. That's just a bit of trial and error. However, we do know that we're in a position where we have an $x \in A$ that we'd like to work with (namely, our goal in this section is to get $x \in B$), so perhaps we should use the fact that $x \in A$ means that we'll have $x \in A \cap B$.

Let's suppose that we do decide to do this. What will that buy us? Well, if we look back to how set intersections work, we'll see that if $x \in A \cap B$ we can conclude that $x \in A$ and $x \in B$. And boy, is that useful! After all, we ultimately wanted to show that $x \in B$.

So that means that we'd expect to take two steps here. First, we'll use the fact that $A = A \cap B$, plus our knowledge that $x \in A$, to get $x \in A \cap B$. From there, we'll expand $x \in A \cap B$ into $x \in A$ and $x \in B$, and then we just need to wrap things up.

Here's what this looks like:

Theorem: For any sets A and B, we have $A \cap B = A$ if and only if $A \in \wp(B)$.

Proof: Let A and B be arbitrary sets. We will prove both directions of implication.

(⇒) First, let's assume that $A \cap B = A$. We will prove that $A \in \wp(B)$. To do so, we'll prove that $A \subseteq B$ by picking an arbitrary $x \in A$ and showing that $x \in B$. Starting with $x \in A$, we'll use the fact that $A = A \cap B$ to conclude that $x \in A \cap B$. Then, since $x \in A \cap B$, we learn that $x \in A$ and $x \in B$. In particular, that means that $x \in B$, which is what we needed to show.

(⇐) Next, assume that $A \in \wp(B)$. We will prove that $A \cap B = A$. To do so, we'll prove that $A \cap B \subseteq A$ and that $A \subseteq A \cap B$. [*the rest of this proof goes here.*]

There's a lot going on in here, so I strongly recommend that you stop reading and go over that new section slowly and carefully to make sure everything seems well-motivated. Not sure what a certain step is doing? No worries! Go ask on Piazza. Once you're satisfied that we've indeed proved one of the two directions of implication, carry on to the next section, where we'll work on the other.

For this other direction of implication, we find ourselves tasked with proving two separate statements: first, that $A \cap B \subseteq A$, and second, that $A \subseteq A \cap B$. Let's take each of these on individually.

We'll begin by proving that $A \cap B \subseteq A$. You might notice that this statement is true about *any* sets A and B. If you're not sure why this is, again, draw a Venn diagram. This one has a pretty intuition. But remember – it's *just* an intuition, and to formalize this proof we're going to need to use the elemental theory of sets and proceed one element at a time. Here's one way we can do this:

> ***Theorem:*** For any sets A and B, we have $A \cap B = A$ if and only if $A \in \wp(B)$.
>
> ***Proof:*** Let A and B be arbitrary sets. We will prove both directions of implication.
>
> ($\Rightarrow$) First, let's assume that $A \cap B = A$. We will prove that $A \in \wp(B)$. To do so, we'll prove that $A \subseteq B$ by picking an arbitrary $x \in A$ and showing that $x \in B$. Starting with $x \in A$, we'll use the fact that $A = A \cap B$ to conclude that $x \in A \cap B$. Then, since $x \in A \cap B$, we learn that $x \in A$ and $x \in B$. In particular, that means that $x \in B$, which is what we needed to show.
>
> ($\Leftarrow$) Next, assume that $A \in \wp(B)$. We will prove that $A \cap B = A$. To do so, we'll prove that $A \cap B \subseteq A$ and that $A \subseteq A \cap B$.
>
> First, we'll prove that $A \cap B \subseteq A$. To do so, pick any $x \in A \cap B$. We will prove that $x \in A$. To do so, notice that since $x \in A \cap B$, we have $x \in A$ and $x \in B$. That specifically means that $x \in A$, as needed.
>
> Next, we'll prove that $A \subseteq A \cap B$. [*the rest of this proof goes here.*]

Take a look over this part of the proof. Notice that everything you're seeing there, from the setup of the proof (showing that one set is a subset of another), to the way in which we expand out definitions (here, what it means for $x \in A \cap B$ to be true), follows the exact same set of rules we've been playing with the whole time. This is a good thing – it means that once you've gotten the patterns down, these sorts of arguments will become a lot easier to work through!

So now we're left with proving $A \subseteq A \cap B$. This statement is not true in general. For example, if I pick A to be the set $\mathbb{N}$ and B to be the set $\varnothing$, then $A \not\subseteq A \cap B$. Oh no! That's not good. But fortunately, that's okay here. Look back at where we are in the proof. We're in the proof of the reverse direction of implication, which means that we're operating under the assumption that $A \in \wp(B)$. As a result, we're not working with just any old pair of sets A and B. We're working with sets where $A \in \wp(B)$.

… which means, what, exactly? Well, look back at the section about power sets. Since $A \in \wp(B)$, we know that $A \subseteq B$. And with that in mind, look back at $A \subseteq A \cap B$. Let's think about this intuitively, for the moment. If you take A and intersect it with B, since $A \subseteq B$, you won't "filter out" any elements of B. There's nothing in A that isn't also in B. So in that sense, the statement $A \subseteq A \cap B$ is a little bit more intuitive. Everything in A (the left-hand side) is still going to be there in $A \cap B$ (the right-hand side).

But of course, that's not the end of the story. That's very much a high-level, intuitive argument as to why $A \subseteq A \cap B$ has to hold here, and we're looking for an elemental set theory explanation. That means that we need to do what we've done a bunch of times before, which is to appeal to the formal definitions. We'll need to pick an arbitrary $x \in A$, use the fact that $A \subseteq B$ to place $x \in B$, and from there recognize that because both $x \in A$ and $x \in B$ that we've got $x \in A \cap B$. Is that a lot of details to check? Yes. But is it difficult? Not really, once you get used to it. Pretty much everything I described here follows from the specific rules about what you can

conclude from different properties of sets holding and what you need to prove in order to show various results about sets.

Converting that sketch of an argument into a formal proof gives us this, the final version of the proof:

Theorem: For any sets A and B, we have $A \cap B = A$ if and only if $A \in \wp(B)$.

Proof: Let A and B be arbitrary sets. We will prove both directions of implication.

($\Rightarrow$) First, let's assume that $A \cap B = A$. We will prove that $A \in \wp(B)$. To do so, we'll prove that $A \subseteq B$ by picking an arbitrary $x \in A$ and showing that $x \in B$. Starting with $x \in A$, we'll use the fact that $A = A \cap B$ to conclude that $x \in A \cap B$. Then, since $x \in A \cap B$, we learn that $x \in A$ and $x \in B$. In particular, that means that $x \in B$, which is what we needed to show.

($\Leftarrow$) Next, assume that $A \in \wp(B)$. We will prove that $A \cap B = A$. To do so, we'll prove that $A \cap B \subseteq A$ and that $A \subseteq A \cap B$.

First, we'll prove that $A \cap B \subseteq A$. To do so, pick any $x \in A \cap B$. We will prove that $x \in A$. To do so, notice that since $x \in A \cap B$, we have $x \in A$ and $x \in B$. That specifically means that $x \in A$, as needed.

Next, we'll prove that $A \subseteq A \cap B$. To do so, consider any $x \in A$; we'll show that $x \in A \cap B$. Earlier, we assumed that $A \in \wp(B)$. This means that $A \subseteq B$, and from this and the fact that $x \in A$ we can conclude that $x \in B$. Collectively, that means we've established $x \in A$ and $x \in B$, so we see that $x \in A \cap B$, which is what we needed to show. ■

And there you go! One complete proof of a theorem about sets.

This particular proof is sneaky in that the statement of the proof is quite short, yet, if you think about it, there are *three* separate statements that all need to be proved independently. First, there's the implication that if $A \cap B = A$, then $A \in \wp(B)$. Then, there's the proof that if A and B are sets, then $A \cap B \subseteq A$. Finally, there's the proof that if $A \in \wp(B)$, then $A \subseteq A \cap B$. The only way we could have known that we needed to do this was to go slowly, methodically, and diligently through the structure of the proof.

You've seen the evolution of this proof, starting purely with a statement of a theorem and ending with all the parts filled in. But imagine we hadn't done that, and that instead we just tossed this proof at you. It would have been difficult to make heads or tails of what the proof was doing, since you're still getting used to writing proofs. You'd have to ask why each step was justified, why there were so many pieces to show, why certain parts weren't "obvious," etc. Our hope was that by having you read through this longer discussion, you can see where each piece of the overall whole comes from, what general patterns we were following, what templates exist for you to build off of, and how much of this larger picture follows from simpler patterns chained together.

In that sense, we hope that reading over this discussion reminds you of learning how to program. Jumping into a full program that solves a problem while you're still learning to code can be extremely disorienting. You'd look at each piece of the program, trying to decipher what on earth it did and how it fit into the overall whole. But in the same way, if you had seen how that program was written, what the thought process was behind each individual piece, and how much of what was written followed from standard techniques, you'd have a better understanding of how everything fit together.

We invite you to look back at this proof when you're studying for the CS103 final exam. How hard was it to read? Did it make sense? Chances are, your answers will be "that wasn't too bad" and "yep, those are all standard techniques." In the meantime, practice working with these templates and combining them in different ways. Dissect the proofs we did in lecture to

see how they adhere to these conventions. And take on the problems from this week's problem set, keeping these techniques in mind. You'll get this. You can do this. Best of luck!